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Hankel singular value functions from Schmidt pairs for nonlinear input–output systems

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Abstract

This paper presents three results in singular value analysis of Hankel operators for nonlinear input–output systems. First, the notion of a Schmidt pair is defined for a nonlinear Hankel operator. This makes it possible to define a Hankel singular value function from a purely input–output point of view and without introducing a state space setting. However, if a state space realization is known to exist then a set of sufficient conditions is given for the existence of a Schmidt pair, and the state space provides a convenient representation of the corresponding singular value function. Finally, it is shown that in a specific coordinate frame it is possible to relate this new singular value function definition to the original state space notion used to describe nonlinear balanced realizations.

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Keywords: Singular values functions; Hankel operators; Adjoint systems; Nonlinear balanced realizations

1. Introduction

Hankel theory for continuous-time nonlinear systems is considerably less developed than its linear counterpart. The classic results are due to Fliess [2,3] who used a system Hankel matrix to describe when an analytic finite-dimensional affine realization of an input–output system described by a Chen–Fliess

functional series is minimal. This matrix in essence plays the same role that the system Hankel matrix does in linear and bilinear system theory [10,11]. In a purely state space setting, the notion of Hankel singular values was generalized to nonlinear systems by Scherpen [13] and applied to model reduction problems. Connections between these invariants and minimality were later described in [15]. A system Hankel operator was introduced in [7,14] for a general nonlinear input–output system and shown to be related, albeit in a fairly weak sense, to the original singular value functions of Scherpen when the input–output operator had a finite-dimensional state space realization. Also

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in a state space setting, [4] describes a notion of eigen-structure for the Hankel operator in terms of the composition of the operator with its Gâteaux derivative.

In this paper three innovations are presented. First the notion of a Schmidt pair is introduced for a nonlinear input–output map. Using this device, it is then possible to define a Hankel singular value function from a purely input–output point of view, i.e., without the need to introduce a state space realization, and without explicitly employing any type of operator differentiation. However, if a finite-dimensional state space realization is known to exist then a set of sufficient conditions is provided for the existence of a Schmidt pair. In particular, it is shown that a state space realization provides a convenient *representation* of a singular value function. Finally, it is shown that in a certain coordinate frame, this new singular value function coincides with the original state space notion found in nonlinear balancing [13]. Therefore, it is believed that this new approach may eventually help solve the nonuniqueness problem for nonlinear balanced realizations reported in [8].

The paper is organized as follows. In Section 2, the nonlinear Hankel operator definition is reviewed in a more general context than it first appeared in [7,14]. In Section 3 a nonlinear extension of a Hilbert adjoint operator is briefly reviewed. This material is essential for understanding how to interpret the generalized Schmidt pair. The new results are all contained in Section 4. The final section applies the theory to a nonlinear spring-damper system.

The mathematical notation used throughout is fairly standard. \mathbb{R}^+ denotes the set of nonnegative real numbers. The inner product and corresponding norm on \mathbb{R}^n are represented, respectively, as $\langle x, y \rangle = x^T y$ and $\|x\| = \sqrt{\langle x, x \rangle}$. $L_p^i[a, b]$ represents the set of Lebesgue measurable functions, i -component vector-valued, with finite L_p norm, $\|\cdot\|_{L_p}$. The inner product on $L_2^i[a, b]$ is denoted by

$$\langle f, g \rangle_{L_2} = \int_a^b f(t)^T g(t) dt.$$

2. Hankel operators induced from input–output systems

Let F be an input–output system defined on a set of admissible inputs $U[t_0, t_1]$ over the time interval

$[t_0, t_1]$. The *time reversal* operator is the injective mapping

$$\begin{aligned} \mathcal{R} : U[t_0, t_1] &\rightarrow U[-t_1, -t_0] \\ u &\rightarrow \hat{u}(t) = u(-t) \end{aligned}$$

and the catenation of two signals $(u, v) \in U[t_0, t_1] \times U[t_2, t_3]$ at $\tau \in [t_0, t_1]$ is defined as

$$(u \#_\tau v)(t) = \begin{cases} u(t) & : t_0 \leq t \leq \tau, \\ v((t - \tau) + t_2) & : \tau < t \leq \tau + (t_3 - t_2). \end{cases}$$

It is generally assumed for any $\tau \in [t_0, t_1]$ that $U[t_0, t_1] = U[t_0, \tau] \#_\tau U[\tau, t_1]$.

Definition 2.1. For any input–output system $F : U[t_0, t_1] \mapsto Y[t_0, t_1]$ with $t_0 < 0 < t_1$, the corresponding *Hankel operator* is

$$\begin{aligned} \mathcal{H}_F : U[0, -t_0] \times U[0, t_1] &\rightarrow Y[0, t_1] \\ (u_-, u_+) &\rightarrow y(t) = F(\underbrace{\mathcal{R}(u_-) \#_0 u_+}_u)(t) \end{aligned}$$

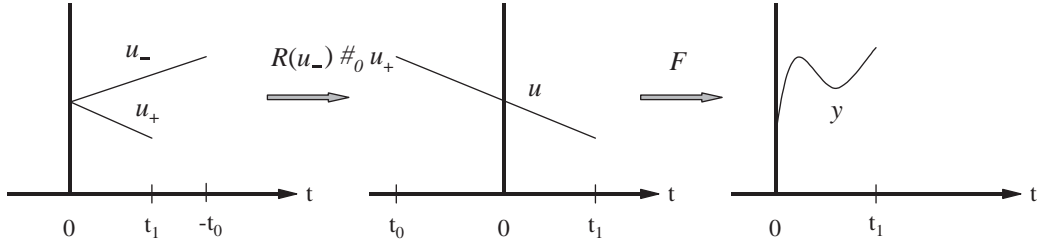
for all $t \in [0, t_1]$. (See Fig. 1.)

The usual interpretation from linear system theory that \mathcal{H}_F maps *past inputs* to *future outputs* is recovered from this definition when F is causal and homogeneous (i.e., $F(0) = 0$). In this context, the zero-input (for positive time) Hankel operator will be denoted by $\mathcal{H}_{F,0}(\hat{u}) = \mathcal{H}_F(\hat{u}, 0)$.

Two inputs $u_-, v_- \in U[0, -t_0]$ are considered equivalent, i.e., $u_- \sim v_-$, when $\mathcal{H}_F(u_-, u_+) = \mathcal{H}_F(v_-, u_+)$ for every $u_+ \in U[0, t_1]$. Each equivalence class under this relation corresponds to the *state* of the system at time $t = 0$. When the quotient set $U[0, -t_0] / \sim$ is locally isomorphic to \mathbb{R}^n then there corresponds an n -dimensional state space realization of F . Our main interest is in operators that have affine input realizations

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, & x(t_0) &= x_0, \\ y &= h(x) \end{aligned} \tag{1}$$

in terms of local coordinates on an n -dimensional state manifold \mathcal{M} . When F is homogeneous, it is always assumed that $f(0) = 0$ and $h(0) = 0$. The existence of any state space realization valid on $[t_0, t_1]$ produces factorizations of \mathcal{H}_F and $\mathcal{H}_{F,0}$ in terms of the

Fig. 1. The Hankel operator, \mathcal{H}_F , corresponding to an input-output mapping F .

controllability and observability operators

$$\begin{aligned} \mathcal{C} : U[0, -t_0] &\mapsto \mathcal{M} \\ :u_- &\mapsto \phi(t, t_0, x_0, \mathcal{R}(u_-))|_{t=0} \end{aligned}$$

and

$$\begin{aligned} \mathcal{O} : U[0, t_1] \times \mathcal{M} &\mapsto Y[0, t_1] \\ :(u_+, x(0)) &\mapsto h(\phi(t, 0, x(0), u_+)), \end{aligned}$$

respectively, where $\phi(t, t_0, x_0, u)$ denotes the solution of the state equation in (1) with $x(t_0) = x_0$ and any admissible input u applied. Specifically, then

$$\mathcal{H}_F(u_-, u_+) = \mathcal{O}(u_+, \mathcal{C}(u_-))$$

and

$$\mathcal{H}_{F,0}(\hat{u}) = \mathcal{O}(0, \mathcal{C}(\hat{u})) =: \mathcal{O}_0 \mathcal{C}(\hat{u}).$$

3. Hilbert adjoints of nonlinear operators

To describe a singular value function of a nonlinear Hankel operator, a generalized Hilbert adjoint operator is needed. It is assumed throughout that the input-output system, F , is L_2 -stable in the sense that $u \in L_2^m(-\infty, 0]$ implies that $F(u)$ restricted to $[0, \infty)$ is in $L_2^\ell[0, \infty)$. In this case, the corresponding zero-input Hankel operator assumes the form $\mathcal{H}_{F,0} : L_2^m[0, \infty) \rightarrow L_2^\ell[0, \infty)$. Viewed as a mapping between Hilbert spaces, it is possible to compute a Hilbert adjoint of $\mathcal{H}_{F,0}$. Various nonlinear extensions of Hilbert adjoints exist in the literature, e.g. most recently [1,9,12]. (A more extensive survey appears in [16].) The following definition, which is fully

developed in [9,16] is most natural for the application considered here.

Definition 3.1. Given two Hilbert spaces H_1 and H_2 , an operator $\mathcal{T} : H_1 \mapsto H_2$ has a global *nonlinear Hilbert adjoint* when there exists an operator $\mathcal{T}^* : H_2 \times H_1 \rightarrow H_1$ such that

$$\begin{aligned} \langle \mathcal{T}(u), y \rangle_{H_2} &= \langle u, \mathcal{T}^*(y, u) \rangle_{H_1}, \\ \forall u \in H_1, \forall y \in H_2, \end{aligned} \quad (2)$$

where $\mathcal{T}^*(y, u)$ is linear in y .

It is often the case that there exists a collection of nontrivial mappings (linear and nonlinear in y) of the form $\mathcal{B} : H_2 \times H_1 \mapsto H_1$ such that $\langle u, \mathcal{B}(y, u) \rangle_{H_1} = 0, \forall u \in H_1, \forall y \in H_2$. In which case, any adjoint mapping \mathcal{T}^* is not uniquely defined since $\mathcal{T}^* + \mathcal{B}$ will also satisfy Eq. (2). In such circumstances, an adjoint operator should be viewed as a member of an equivalence class where two such operators \mathcal{T}^* and $\mathcal{T}^{*'}$ are equivalent when

$$\begin{aligned} \langle u, \mathcal{T}^*(y, u) \rangle_{H_1} &= \langle u, \mathcal{T}^{*'}(y, u) \rangle_{H_1}, \\ \forall u \in H_1, \forall y \in H_2. \end{aligned} \quad (3)$$

A shorthand notation for (3) is simply $\mathcal{T}^*(y, u) = \mathcal{T}^{*'}(y, u)$. Thus, any equality involving adjoint operators really means that both expressions belong to the same equivalence class. It is not necessary in many applications to have a globally defined \mathcal{T}^* . The following theorem provides a sufficient condition for the existence of a locally defined adjoint operator.

Theorem 3.1 (Gray and Scherpen [9], Scherpen and Gray [16]). Suppose H_1 and H_2 are two Hilbert spaces and $U \subset H_1$ is any convex neighborhood of 0. Let $\mathcal{T} : U \mapsto H_2$ be a continuously Fréchet

differentiable mapping on U such that $\mathcal{T}(0) = 0$. Then the mapping

$$\mathcal{T}^*(y, u) = \int_0^1 (D\mathcal{T}(tu))^*(y) dt$$

is a suitable Hilbert adjoint of \mathcal{T} on $H_2 \times U$, where $D\mathcal{T}$ is the Fréchet derivative of \mathcal{T} , and $(\cdot)^*$ denotes the usual linear adjoint operator.

While a useful device in many circumstances, a non-linear Hilbert adjoint operator does not share all of the familiar properties associated with linear adjoints. For example, the sense in which operators can be composed when adjoint operators are present is more complicated since the domain of an adjoint operator is not simply the codomain of the original operator. For example, consider the Hilbert spaces H_i , $i = 1, 2, 3$, the operators

$$\begin{aligned} \mathcal{T}: H_1 &\mapsto H_2, & \mathcal{S}: H_2 &\mapsto H_3 \\ :u &\mapsto w & :w &\mapsto y \end{aligned}$$

and the corresponding adjoints

$$\begin{aligned} \mathcal{T}^*: H_2 \times H_1 &\mapsto H_1, & \mathcal{S}^*: H_3 \times H_2 &\mapsto H_2 \\ :(w, u) &\mapsto \bar{u} & :(y, w) &\mapsto \bar{w}. \end{aligned}$$

Clearly the composition and its adjoint

$$\begin{aligned} \mathcal{S}\mathcal{T}: H_1 &\mapsto H_3, & (\mathcal{S}\mathcal{T})^*: H_3 \times H_1 &\mapsto H_1 \\ :u &\mapsto y & :(y, u) &\mapsto \bar{u}. \end{aligned}$$

are well defined, but no *direct* composition like $\mathcal{T}^*\mathcal{T}$ or $\mathcal{T}^*\mathcal{S}^*$ is possible as in the classic setting. Still some *formal* compositions can be defined which have great utility in a variety of situations.

Definition 3.2. Let H_i , $i = 1, 2, 3$, be a collection of Hilbert spaces. Assume $\mathcal{T}: H_1 \mapsto H_2$ and $\mathcal{S}: H_2 \mapsto H_3$ are two operators with well-defined adjoint operators. Define the following operator products:

$$\begin{aligned} (\mathcal{S}^*\mathcal{T})_1: H_1 \times H_2 &\mapsto H_2 \quad [\text{when } H_2 = H_3] \\ :(u, w) &\mapsto \mathcal{S}^*(\mathcal{T}(u), w) \end{aligned}$$

$$\begin{aligned} (\mathcal{S}^*\mathcal{T})_2: H_3 \times H_1 &\mapsto H_1 \\ :(y, u) &\mapsto \mathcal{S}^*(y, \mathcal{T}(u)). \end{aligned}$$

Of particular interest in the next section is the self-adjoint operator $\mathcal{H}_{F,0}^* \mathcal{H}_{F,0}(u) := (\mathcal{H}_{F,0}^* \mathcal{H}_{F,0})_1(u, u)$. It forms the basis of our singular value function analysis.

4. Hankel singular value functions from Schmidt pairs

The notion of a singular value function is first developed in a *coordinate free* setting. This is accomplished by defining a Schmidt pair for the operator $\mathcal{H}_{F,0}$. Let $\pi: L_2^m[0, \infty) \mapsto L_2^m[0, \infty)/\sim$ denote the canonical projection induced by $\mathcal{H}_{F,0}$. For any nonzero function $\hat{v} \in L_2^m[0, \infty)$ and real numbers $0 < a < 1$, $b > 0$ define $V(a, b) = \{\hat{v}_\varepsilon = \varepsilon \hat{v}: \varepsilon \in (1-a, 1+b)\}$.

Definition 4.1. A *Schmidt pair* (\hat{v}, \mathcal{U}) for a Hankel operator $\mathcal{H}_{F,0}: L_2^m[0, \infty) \mapsto L_2^\ell[0, \infty]$ and some given adjoint operator $\mathcal{H}_{F,0}^*$ is a nonzero function $\hat{v} \in L_2^m[0, \infty)$ and an operator $\mathcal{U}: V(a, b) \mapsto L_2^\ell[0, \infty]$ such that

$$\begin{aligned} \mathcal{H}_{F,0}(\hat{v}_\varepsilon) &= \sigma(\pi(\hat{v}_\varepsilon))\mathcal{U}(\hat{v}_\varepsilon), \\ \mathcal{H}_{F,0}^*(\mathcal{U}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) &= \sigma(\pi(\hat{v}_\varepsilon))\hat{v}_\varepsilon, \end{aligned}$$

for all $\hat{v}_\varepsilon \in V(a, b)$ and some function $\sigma: V(a, b)/\sim \mapsto \mathbb{R}^+$.

When such a pair (\hat{v}, \mathcal{U}) exists, the linearity of $\mathcal{H}_{F,0}^*$ in its first argument implies directly that

$$\mathcal{H}_{F,0}^*(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) = \sigma^2(\pi(\hat{v}_\varepsilon))\hat{v}_\varepsilon.$$

Therefore, σ is logically called a *singular value function* for the operator pair $(\mathcal{H}_{F,0}, \mathcal{H}_{F,0}^*)$. Singular values functions are strongly dependent on the choice of adjoint operator. For example, if $\mathcal{H}_{F,0}'$ is a second adjoint operator distinct from $\mathcal{H}_{F,0}^*$ then

$$\begin{aligned} \mathcal{H}_{F,0}'(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) &= \sigma^2(\pi(\hat{v}_\varepsilon))\hat{v}_\varepsilon \\ &\quad + \sigma(\pi(\hat{v}_\varepsilon))\mathcal{B}(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) \end{aligned}$$

for some function $\mathcal{B}: L_2^\ell[0, \infty) \times L_2^m[0, \infty) \mapsto L_2^m[0, \infty)$ with $\langle \hat{v}_\varepsilon, \mathcal{B}(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) \rangle_{L_2^m} = 0$ everywhere on $V(a, b)$. So clearly σ is not a singular value function for the pair $(\mathcal{H}_{F,0}, \mathcal{H}_{F,0}')$. Different adjoint operators can therefore potentially produce different

singular value functions. But they all share the property that

$$\langle \hat{v}_\varepsilon, \mathcal{H}_{F,0}^*(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) \rangle_{L_2} = \sigma^2(\pi(\hat{v}_\varepsilon)) \langle \hat{v}_\varepsilon, \hat{v}_\varepsilon \rangle_{L_2}.$$

For a linear operator, $V(a, b)$ is normally taken as the span of \hat{v} over \mathbb{R} with $\|\hat{v}\|_{L_2} = 1$. For compact linear operators, constant singular values functions and linear \mathcal{U} operators are known to always exist. In fact, the operator $\mathcal{H}_{F,0}$ has a singular value decomposition of the form

$$\mathcal{H}_{F,0}(\hat{u}) = \sum_{i=1}^{\infty} \sigma_i \mathcal{U}_i(\hat{u}) \langle \hat{v}_i, \hat{u} \rangle_{L_2}, \quad \forall \hat{u} \in L_2^m[0, \infty),$$

where each $(\hat{v}_i, \mathcal{U}_i)$ is a Schmidt pair, $\sigma_i \geq \sigma_{i+1}$ for $i \geq 1$, and $\{\hat{v}_i\}_{i=1}^{\infty}$ is a complete orthonormal set for $L_2^m[0, \infty)$. In the nonlinear setting, when a family of Schmidt pairs $\{(\hat{v}_i, \mathcal{U}_i)\}_{i=1}^{\infty}$ is known to exist, the analogous expression is

$$\mathcal{H}_{F,0}(\hat{u}) = \sum_{i=1}^{\infty} \sigma_i(\pi(\hat{u})) \mathcal{U}_i(\hat{u}) \langle \hat{v}_i, \hat{u} \rangle_{L_2}, \quad \forall \hat{u} \in V,$$

where V is a subset of $L_2^m[0, \infty)$ at least containing each $V_i(a_i, b_i)$. Also, unlike the linear case, this decomposition will be highly nonunique when the set of adjoint operators for $\mathcal{H}_{F,0}$ is large. Thus, distinct decompositions truncated to the same number of leading terms will result in different approximations of $\mathcal{H}_{F,0}$. This has obvious consequences for any nonlinear model reduction algorithm based on singular value functions (see [8] for a related discussion).

When F is homogeneous with a smooth n -dimensional state space realization $(f, g, h, 0)$, which is L_2 input-to-state stable on a neighborhood W of 0 (which means that when $u \in L_2^m(-\infty, 0]$, the corresponding state vector, $x(t)$, assuming the initial condition $x(-\infty) = 0$, is finite on $(-\infty, 0]$ and always contained in W), it is possible to prove the existence of n Schmidt pairs and singular value functions for $\mathcal{H}_{F,0}$. The state space context also provides a convenient representation for these functions. This is accomplished using the *energy functions* for $(f, g, h, 0)$ as described below.

Definition 4.2. The *controllability* and *observability functions* for the system $(f, g, h, 0)$ are defined,

respectively, as

$$L_c(x) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0, \quad x(0)=x}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt$$

and

$$L_o(x) = \frac{1}{2} \int_0^{\infty} \|y(t)\|^2 dt,$$

when $x(0) = x$, and $u(t) = 0$ for $0 \leq t < \infty$.

The following result is known.

Theorem 4.1 (Scherpen [13]). *Consider a system $(f, g, h, 0)$ where*

- (A1) *f is asymptotically stable on some neighborhood Y of 0;*
- (A2) *The system (f, g, h) is zero-state observable on Y (i.e., $\mathcal{O}_0(x_0) \equiv 0$ implies that $x_0 = 0$);*
- (A3) *L_c and L_o exist and are smooth on Y .*

There exists a coordinate transformation $x = \psi(z)$, $\psi(0)=0$, defined on a neighborhood U of 0 which converts the system into an input-normal/output-diagonal realization, where

$$\tilde{L}_c(z) := L_c(\psi(z)) = \frac{1}{2} z^T z,$$

$$\tilde{L}_o(z) := L_o(\psi(z)) = \frac{1}{2} z^T \text{diag}(\tau_1(z), \dots, \tau_n(z)) z$$

with $\tau_1(z) \geq \dots \geq \tau_n(z)$ being smooth functions on $W := \psi^{-1}(U)$ provided the number of distinct $\tau_i(z)$'s are constant over W .

The set of functions $\tau_i, i=1, \dots, n$ are called *singular value functions* of $(f, g, h, 0)$ in [13]. They should not be confused with singular value functions, σ_i , for $(\mathcal{H}_{F,0}, \mathcal{H}_{F,0}^*)$, though as will be shown momentarily, there is a relationship between the two concepts. When \tilde{L}_o is not in a diagonal form, the realization is said to simply be in *input-normal form*. It is also known that there exists a coordinate transformation $\bar{z} = \eta(z)$, $\eta(0) = 0$, defined on a neighborhood of 0 which converts the system into a *balanced realization*,

where

$$\begin{aligned}\check{L}_c(\bar{z}) &:= \check{L}_c(\eta^{-1}(\bar{z})) \\ &= \frac{1}{2}\bar{z}^T \text{diag}(\check{\tau}_1(\bar{z}_1)^{-1}, \dots, \check{\tau}_n(\bar{z}_n)^{-1})\bar{z},\end{aligned}$$

$$\begin{aligned}\check{L}_o(\bar{z}) &:= \check{L}_o(\eta^{-1}(\bar{z})) \\ &= \frac{1}{2}\bar{z}^T \text{diag}(\check{\tau}_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})), \dots, \\ &\quad \check{\tau}_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})))\bar{z},\end{aligned}$$

with $\check{\tau}_i(\bar{z}_i) := \tau_i(0, \dots, 0, \eta_i^{-1}(\bar{z}_i), 0, \dots, 0)^{1/2}$ for $i = 1, \dots, n$. Along coordinate axes it is easily verified that

$$\begin{aligned}\check{L}_c(0, \dots, 0, \bar{z}_i, 0, \dots, 0) &= \frac{1}{2}\bar{z}_i^2 \check{\tau}_i(\bar{z}_i)^{-1}, \\ \check{L}_o(0, \dots, 0, \bar{z}_i, 0, \dots, 0) &= \frac{1}{2}\bar{z}_i^2 \check{\tau}_i(\bar{z}_i).\end{aligned}$$

To relate the singular values functions of $(\mathcal{H}_{F,0}, \mathcal{H}_{F,0}^*)$ to those of a given state space realization of F , the key idea is to select the adjoint operator for $\mathcal{H}_{F,0}$ in a manner *consistent* with the realization. This is done in the following two theorems. The first theorem is adapted from [5,6]. It expresses an adjoint operator in terms of a solution to a two-point boundary value problem with conjugate points $t_0 = -\infty$ and $t_1 = \infty$. Its original proof is done by viewing the system as a port-controlled Hamiltonian system. The second theorem applies the first theorem. It provides sufficient conditions for the existence of a Schmidt pair using the particular adjoint operator described below.

Theorem 4.2. *Let F be a causal homogeneous L_2 -stable input–output mapping with a smooth n -dimensional state space realization $\Sigma = (f, g, h, 0)$ that is L_2 input-to-state stable on a neighborhood W of 0. Consider the mapping*

$$\begin{aligned}\mathcal{H}_{\Sigma,0}^* : L_2^\ell[0, \infty) \times L_2^m[0, \infty) &\mapsto L_2^m[0, \infty) \\ : (u_a, \hat{u}) &\mapsto y_a\end{aligned}$$

defined by the state space realization

$$\dot{z} = f(z) + g(z)\mathcal{R}_-(\hat{u}), \quad z(-\infty) = 0, \quad (4)$$

$$\dot{p} = -A^T(z)p - C^T(z)u_a, \quad p(\infty) = 0, \quad (5)$$

$$y_a = \mathcal{R}_+(g^T(z)p) \quad (6)$$

with

$$\begin{aligned}\mathcal{R}_+(u) &= \begin{cases} u(-t) & t \in [0, \infty), \\ 0 & t \in (-\infty, 0], \end{cases} \\ \mathcal{R}_-(\hat{u}) &= \begin{cases} 0 & t \in [\infty, 0], \\ \hat{u}(-t) & t \in (-\infty, 0], \end{cases}\end{aligned}$$

$A(z) \in \mathbb{R}^{n \times n}$, $C(z) \in \mathbb{R}^{\ell \times n}$ such that $f(z) = A(z)z$ and $h(z) = C(z)z$ for all $z \in W$, and where it is assumed a priori that $p(-\infty)$ is finite for all $(u_a, \hat{u}) \in L_2^\ell[0, \infty) \times L_2^m[0, \infty)$. (By assumption, $u_a(t) = 0$ when $t \leq 0$.) Then $\mathcal{H}_{\Sigma,0}^*$ is a valid Hilbert adjoint of $\mathcal{H}_{F,0}$. That is,

$$\langle \mathcal{H}_{F,0}(\hat{u}), u_a \rangle_{L_2} = \langle \hat{u}, \mathcal{H}_{\Sigma,0}^*(u_a, \hat{u}) \rangle_{L_2},$$

for all $u_a \in L_2^\ell[0, \infty)$ and $\hat{u} \in L_2^m[0, \infty)$.

It should be noted that the factorizations $f(z) = A(z)z$ and $h(z) = C(z)z$ are always possible since it is assumed that $f(0) = 0$ and $h(0) = 0$. But, as is well known, these factorizations are not unique. This means that potentially a set of consistent adjoints for $\mathcal{H}_{F,0}$ is possible. The L_2 input-to-state stability of Σ and the assumption that $p(-\infty)$ is finite insures that the state equation for $\mathcal{H}_{\Sigma,0}^*$ has a well defined solution for all time and every admissible input.

Theorem 4.3. *Let $\Sigma = (f, g, h, 0)$ be a smooth L_2 input-to-state stable realization in input-normal form of a causal homogeneous L_2 -stable input–output mapping F on a neighborhood W of 0. Let \bar{z} be a fixed nonzero state in W and $\{\bar{z}_\varepsilon = \varepsilon\bar{z} : \varepsilon \in (1-a, 1+b), 0 < a < 1, b > 1\}$ a line segment in W . Let v_ε denote the minimum energy input which drives $z(-\infty) = 0$ to $z(0) = \bar{z}_\varepsilon$, and $\hat{v}_\varepsilon = \mathcal{R}_+(v_\varepsilon)$. Suppose the following assumptions are valid:*

- (A1) f is asymptotically stable on W ;
- (B1) There exists a factorization $f(z) = A(z)z$, where the symmetric part $A_s(z) := (A(z) + A^T(z))/2 = -g(z)g^T(z)/2$ for all $z \in W$;
- (B2) System (4) and (5) with $(u_a, \hat{u}) = (\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon)$ has a well-defined solution for any $\varepsilon \in (1-a, 1+b)$ with the property that $p(0) = \sigma^2(\bar{z}_\varepsilon)\bar{z}_\varepsilon$ for some positive real number $\sigma^2(\bar{z}_\varepsilon)$.

Then the operator pair $(\mathcal{H}_{F,0}, \mathcal{H}_{\Sigma,0}^*)$ has a Schmidt pair with corresponding singular value function $\sigma \circ \pi$ on $V(a, b) = \{\hat{v}_\varepsilon : \varepsilon \in (1-a, 1+b)\}$.

Proof. The proof is constructive. Define $\hat{v} = \hat{v}_\varepsilon|_{\varepsilon=1}$ and

$$\begin{aligned}\mathcal{U}: V(a, b) &\mapsto L_2^\ell[0, \infty) \\ \hat{v}_\varepsilon &\mapsto \mathcal{H}_{F,0}(\hat{v}_\varepsilon)/\sigma(\tilde{z}_\varepsilon).\end{aligned}$$

The claim is that (\hat{v}, \mathcal{U}) is a Schmidt pair for $(\mathcal{H}_{F,0}, \mathcal{H}_{\Sigma,0}^*)$. By design, $\mathcal{H}_{F,0}(\hat{v}_\varepsilon) = \sigma(\tilde{z}_\varepsilon)\mathcal{U}(\hat{v}_\varepsilon) = \sigma(\pi(\hat{v}_\varepsilon))\mathcal{U}(\hat{v}_\varepsilon)$ for all $\hat{v}_\varepsilon \in V(a, b)$. To verify the rest of the definition, first recall that in [13] it was shown that (A1) implies that $\tilde{L}_c(z) = \frac{1}{2}z^T z$ is the smooth anti-stabilizing solution of the partial differential equation

$$\frac{\partial \tilde{L}_c}{\partial z} f(z) + \frac{1}{2} \frac{\partial \tilde{L}_c}{\partial z} g(z) g^T(z) \frac{\partial \tilde{L}_c}{\partial z} = 0. \quad (7)$$

Furthermore, $v_\varepsilon = g^T(z) \partial \tilde{L}_c / \partial z = g^T(z) z$ when evaluated along the solution of $\dot{z} = f(z) + g(z)v$ starting at $z(-\infty) = 0$ and terminating at $z(0) = \tilde{z}_\varepsilon$. Now setting $u_a = \mathcal{H}_{F,0}(\hat{v}_\varepsilon)$ and $\hat{u} = \hat{v}_\varepsilon$, the realization of $\mathcal{H}_{\Sigma,0}^*$ evaluated at these inputs has the equivalent form for $t \leq 0$:

$$\dot{z} = f(z) + g(z)g^T(z)z, \quad z(0) = \tilde{z}_\varepsilon, \quad (8)$$

$$\dot{p} = -A^T(z)p, \quad p(0) = \tilde{p}_\varepsilon, \quad (9)$$

for some $\tilde{p}_\varepsilon \in \mathbb{R}^n$. Eq. (7) reduces to

$$z^T[A(z) + A^T(z) + g(z)g^T(z)]z = 0.$$

This does *not* in general imply that $A(z) + A^T(z) + g(z)g^T(z) = 0$, but if this is the case, i.e., if assumption (B1) is satisfied, then Eqs. (8) and (9) become

$$\dot{z} = (A(z) + g(z)g^T(z))z, \quad z(0) = \tilde{z}_\varepsilon, \quad (10)$$

$$\dot{p} = (A(z) + g(z)g^T(z))p, \quad p(0) = \tilde{p}_\varepsilon. \quad (11)$$

Now if \tilde{p}_ε and \tilde{z}_ε are related by the constant $\sigma^2(\tilde{z}_\varepsilon)$, as per assumption (B2), then Eqs. (10) and (11) will have solutions that are related by this same constant for all negative time. That is, $p(t) = \sigma^2(\tilde{z}_\varepsilon)z(t)$ when $t \leq 0$. Hence,

$$\begin{aligned}\mathcal{H}_{\Sigma,0}^*(\mathcal{H}_{F,0}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) &= \mathcal{R}_+(g^T(z)p) \\ &= \mathcal{R}_+(g^T(z)\sigma^2(\tilde{z}_\varepsilon)z) \\ &= \sigma^2(\tilde{z}_\varepsilon)\mathcal{R}_+(g^T(z)z) \\ &= \sigma^2(\tilde{z}_\varepsilon)\mathcal{R}_+(v_\varepsilon) \\ &= \sigma^2(\tilde{z}_\varepsilon)\hat{v}_\varepsilon.\end{aligned}$$

Using the linearity of the first argument of $\mathcal{H}_{\Sigma,0}^*$, the desired result then follows:

$$\begin{aligned}\mathcal{H}_{\Sigma,0}^*(\mathcal{U}(\hat{v}_\varepsilon), \hat{v}_\varepsilon) &= \sigma(\tilde{z}_\varepsilon)\hat{v}_\varepsilon \\ &= \sigma(\pi(\hat{v}_\varepsilon))\hat{v}_\varepsilon. \quad \square\end{aligned}$$

The factorization property in (B1) is automatically satisfied in the linear setting because for any (A, B) in input normal form, the Lyapunov equation $A + A^T + BB^T = 0$ is always satisfied. But in the nonlinear case not much is known about these types of factorizations. (A related factorization is described and characterized in [8].) Fortunately, the boundary property in (B2) can be assured when the realization is in the more refined input-normal/output-diagonal form. In addition, the next theorem shows that in such a coordinate frame, the singular value functions defined for a Schmidt pair will coincide with the singular value functions defined in Theorem 4.1 when each is evaluated along a coordinate axis.

Theorem 4.4. *Let $\Sigma = (f, g, h, 0)$ be a smooth L_2 input-to-state stable realization in input-normal/output-diagonal form of a causal homogeneous L_2 -stable input-output mapping F in a neighborhood W of 0. Assume τ_i , $i = 1, \dots, n$ are singular value functions of Σ . For any $i \in \{1, 2, \dots, n\}$, let $\tilde{z}_i = (0, \dots, 0, z_i, 0, \dots, 0)^T$ be a fixed nonzero state in W and $\{\tilde{z}_{i,\varepsilon} = \varepsilon \tilde{z}_i; \varepsilon \in (1-a_i, 1+b_i), 0 < a_i < 1, b_i > 1\}$ a line segment in W . Let $V_i(a_i, b_i)$ denote the corresponding set of minimum energy inputs. Suppose the following assumptions are valid:*

(A1) *f is asymptotically stable on W .*

(B1) *There exists a factorization $f(z) = A(z)z$, where the symmetric part $A_s(z) := (A(z) + A^T(z))/2 = -g(z)g^T(z)/2$ for all $z \in W$.*

(C1) *The system (4) and (5) with $(u_a, \hat{u}) = (\mathcal{H}_{F,0}(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon})$ has a well defined nontrivial (i.e., nonzero) solution for every $i \in \{1, \dots, n\}$ and $\varepsilon \in (1-a_i, 1+b_i)$.*

Then it follows that the operator pair $(\mathcal{H}_{F,0}, \mathcal{H}_{\Sigma,0}^*)$ has n Schmidt pairs with corresponding singular value functions $\tau_i^{1/2} \circ \pi$ on $V_i(a_i, b_i)$, $i = 1, \dots, n$.

Proof. In the context of the proof of Theorem 4.3, let $(u_a, \hat{u}) = (\mathcal{H}_{F,0}(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon})$ for any $i \in \{1, \dots, n\}$ and $\hat{v}_{i,\varepsilon} \in V_i(a_i, b_i)$. Then Eqs. (4) and (5) become

for $t > 0$:

$$\dot{z} = f(z), \quad z(0) = \tilde{z}_{i,\varepsilon},$$

$$\dot{p} = -A^T(z)p - c^T(z)h(z), \quad p(0) = \tilde{p}_{i,\varepsilon},$$

where $\tilde{p}_{i,\varepsilon}$ is well defined, as per assumption (C1), but unspecified for the moment (in contrast to the situation in Theorem 4.3). For any $z \in W$ it follows that

$$z^T \dot{p} = -f^T(z)p - h^T(z)h(z),$$

or equivalently,

$$\frac{d(z^T p)}{dt} = -h^T(z)h(z).$$

Therefore, integrating both sides of the equation over the trajectory of $(z(t), p(t))$ from $t = 0$ to $t = \infty$ with $(z(0), p(0)) = (\tilde{z}_{i,\varepsilon}, \tilde{p}_{i,\varepsilon})$ and $(z(\infty), p(\infty)) = (0, 0)$ produces

$$\begin{aligned} \frac{1}{2} \tilde{z}_{i,\varepsilon}^T \tilde{p}_{i,\varepsilon} &= \tilde{L}_o(\tilde{z}_{i,\varepsilon}) \\ &= \frac{1}{2} \tau_i(\tilde{z}_{i,\varepsilon})(\varepsilon z_i)^2. \end{aligned}$$

Since $z_i \neq 0$, selecting the boundary condition $\tilde{p}_{i,\varepsilon} = \tau_i(\tilde{z}_{i,\varepsilon})\tilde{z}_{i,\varepsilon}$ at $t = 0$ will insure that the operator pair has a well defined Schmidt pair, and in fact, the corresponding singular value function must satisfy $\sigma_i^2(\pi(\hat{v}_{i,\varepsilon})) = \sigma_i^2(\tilde{z}_{i,\varepsilon}) = \tau_i(\tilde{z}_{i,\varepsilon}) = \tau_i(\pi(\hat{v}_{i,\varepsilon}))$. \square

5. Example: nonlinear spring-damper system

Consider the forced spring-damper system shown in Fig. 2, which is described by the Duffing equation $m\ddot{d} + c\dot{d} + k_1 d + k_2 d^3 = \sqrt{2c}u$, where d denotes the displacement from the equilibrium position $d = 0$, and u is an applied force. Define the states $x_1 = d$ and $x_2 = m\dot{d}$ and select the output function $y = \sqrt{2c}x_2/m$. This Hamiltonian system has a state space realization

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \frac{1}{m}x_2 \\ -k_1 x_1 - k_2 x_1^3 - \frac{c}{m}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{2c} \end{bmatrix} u, \\ y &= \frac{\sqrt{2c}}{m} x_2 \end{aligned}$$

with corresponding energy functions $L_c(x) = L_o(x) = \frac{1}{2} x^T \text{diag}(k_1 + k_2 x_1^2/2, 1/m)x$. Applying the

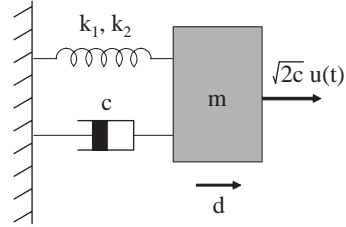


Fig. 2. Nonlinear spring-damper system.

coordinate transformation

$$z = \psi^{-1}(x) = \begin{bmatrix} \sqrt{k_1 + \frac{1}{2}k_2 x_1^2} x_1 \\ \frac{1}{\sqrt{m}} x_2 \end{bmatrix}$$

about a neighborhood of $x = 0$ produces the corresponding balanced realization

$$\dot{z} = f(z) + g(z)u = \begin{bmatrix} \alpha(z) \frac{z_2}{z_1} \\ -\alpha(z) - \frac{c}{m} z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{\frac{2c}{m}} \end{bmatrix} u,$$

$$y = h(z) = \sqrt{\frac{2c}{m}} z_2,$$

where

$$\alpha(z) := \frac{1}{\sqrt{m}} (k_1 + k_2 x_1^2) x_1 \Big|_{x=\psi(z)}$$

and the transformed energy functions are $\tilde{L}_c(z) = \tilde{L}_o(z) = \frac{1}{2} z^T z$. In this case, the singular value functions of the realization are $\tau_1(z) = \tau_2(z) = 1$. Theorem 4.4 applies in this coordinate frame provided that $F : u \mapsto y$ has the stated properties and assumptions (A1)–(C1) are satisfied. It can be directly verified that F is homogeneous and input-to-state stable, and that f is globally asymptotically stable about $z = 0$. In addition, the factorization

$$f(z) = A(z)z = \begin{bmatrix} 0 & \frac{\alpha(z)}{z_1} \\ -\frac{\alpha(z)}{z_1} & -\frac{c}{m} \end{bmatrix} z$$

satisfies (B1). It will be shown empirically that (C1) is satisfied for the $A(z)$ above and $C(z) = [0 \ \sqrt{2c/m}]$.

According to Theorem 4.4, the corresponding operator pair $(\mathcal{H}_{F,0}, \mathcal{H}_{\Sigma,0}^*)$, where $\Sigma = (f, g, h, 0)$, has (at least) two Schmidt pairs: $(\hat{v}_1, \mathcal{U}_1)$ and $(\hat{v}_2, \mathcal{U}_2)$, one for each coordinate direction. Setting $\tilde{z}_1 = e_1 = [1 \ 0]^T$ and $\tilde{z}_2 = e_2 = [0 \ 1]^T$ provides that $\|\hat{v}_i\|_{L_2}^2 = 2\tilde{L}_c(\tilde{z}_i) = 1$ for $i = 1, 2$. Consider the lightly damped system where $m = 3$, $k_1 = 1$, $k_2 = 4$, and $c = 0.5$. The signals \hat{v}_1

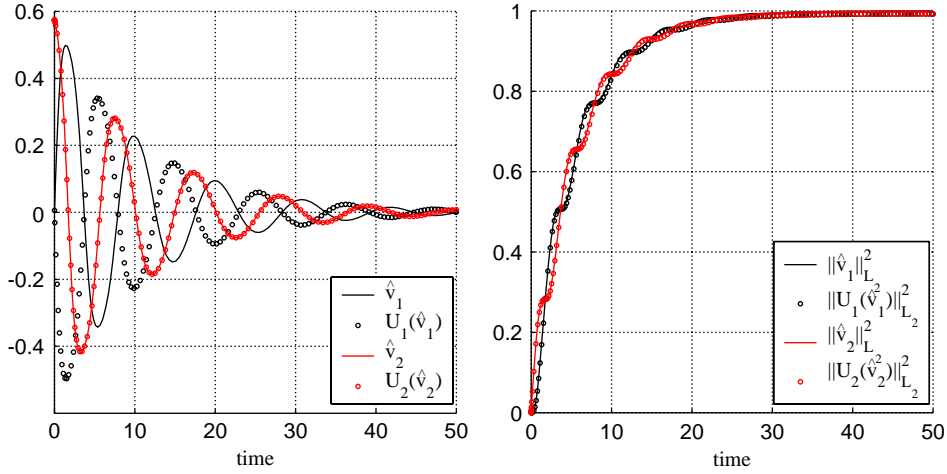


Fig. 3. The Schmidt pair functions $(\hat{v}_i, \mathcal{U}_i(\hat{v}_i))$ and their energy content for the nonlinear spring-damper system.

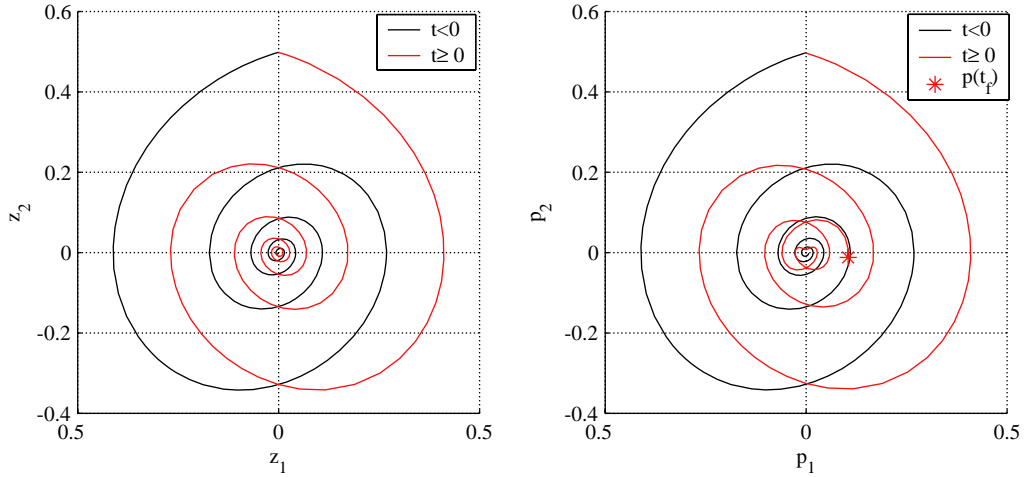


Fig. 4. The z and p trajectories for the adjoint realization when $i = 2$ and $\varepsilon = 0.5$.

and \hat{v}_2 were determined numerically by computing the optimal controls which drive the state of the system $\tilde{\Sigma}$ in reverse time from their respective coordinate axes to the origin over a long interval of time, in this case $t_f = 50$ s was sufficient. These functions are shown in Fig. 3, as well as the corresponding functions $\mathcal{U}_i(\hat{v}_i) = \mathcal{H}_{F,0}(\hat{v}_i)$ and their energy content as function of time. If each $(\hat{v}_i, \mathcal{U}_i)$ forms a Schmidt pair, then for this example it should be the case that $\mathcal{H}_{\Sigma,0}^*(\mathcal{U}(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon}) = \hat{v}_{i,\varepsilon}$ for $i = 1, 2$ and all ε in an open interval containing 1. The adjoint mapping was

implemented using the realization (4)–(6) and the factorizations $f(z) = A(z)z$ and $h(z) = C(z)z$ above. The two point boundary value problem was solved numerically (thus showing that (C1) holds for at least the ε under consideration) by first performing a local search about $p = 0$ to determine the initial condition $p(-t_f)$ which will render $p(0) = \tilde{p}_{i,\varepsilon} = \varepsilon e_i$. Theoretically, this will also insure that $p(t)$ is steered to the origin as $t \mapsto \infty$ by the active input u_a . But numerically, since the equilibrium $p = 0$ is not stable and u_a diminishes for large t , finite precision calculations

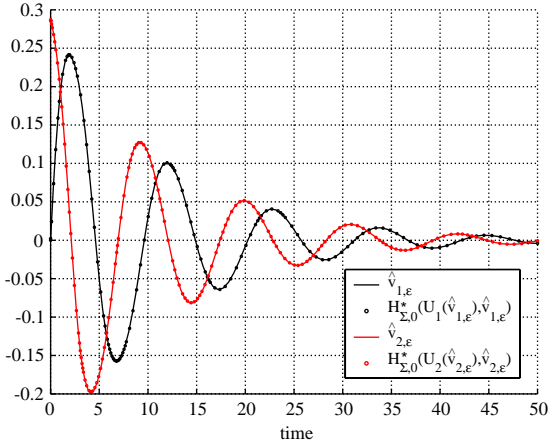


Fig. 5. The functions $\hat{v}_{i,\varepsilon}$ and $\mathcal{H}_{\Sigma,0}^*(\mathcal{U}_i(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon})$ when $\varepsilon = 0.5$.

produce the situation where $p(t)$ misses the origin after some large but finite $t = t^*$ and starts to diverge. Since one is interested in y_a only for negative time, however, this inaccuracy is of little consequence in most cases. Also, to avoid numerical sensitivity near the z_1 coordinate axis, it was particularly useful to employ the expression $\alpha(z)/z_1 = (2\beta(z)/\sqrt{m}) - (k_1/\beta(z))$, where $\beta(z) = \sqrt{k_1 + k_2 x_1^2/2}|_{x=\psi(z)}$. An example of the z and p dynamics is shown in Fig. 4 when $i = 2$ and $\varepsilon = 0.5$. Note here that $p(-t_f) = (6.96, 1.21) \times 10^{-3}$ produces $p(t^*) = p(33.79) = (5.92, 76.12) \times 10^{-4}$ and $p(t_f) = (0.105, -0.012)$. The corresponding signals $\hat{v}_{i,\varepsilon}$ and $\mathcal{H}_{\Sigma,0}^*(\mathcal{U}_i(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon})$ are shown in Fig. 5. As expected, $\hat{v}_{i,\varepsilon}$ and $\mathcal{H}_{\Sigma,0}^*(\mathcal{U}_i(\hat{v}_{i,\varepsilon}), \hat{v}_{i,\varepsilon})$ coincide perfectly despite the numerical inaccuracies. This was the case in this example for every $\varepsilon > 0$ tested.

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